On an unified framework for approachability in games with or without signals

Vianney Perchet [*]and M. Quincampoix[†]

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Abstract

We unify standard frameworks for approachability both in full or partial monitoring by defining a new abstract game, called *the purely informative game*, where the outcome at each stage is the maximal information players can obtain, represented as some probability measure. Objectives of players can be rewritten as the convergence (to some given set) of sequences of averages of these probability measures. We obtain new results extending the approachability theory developed by Blackwell moreover this new abstract framework enables us to characterize approachable sets with, as usual, a remarkably simple and clear reformulation for convex sets.

Translated into the original games, those results become the first necessary and sufficient condition under which an arbitrary set is approachable and they cover and extend previous known results for convex sets. We also investigate a specific class of games where, thanks to some unusual definition of averages and convexity, we again obtain a complete characterization of approachable sets along with rates of convergence.

Introduction Repeated games can be studied by considering sequences of payoffs and constructing, stage by stage, strategies with the requirement that the outcome at the next stage will have good properties given the past ones. Perhaps, the most revealing examples of this claim are Shapley's [18] operator that describes the value of stochastic zero-sum games, the *exponential weight algorithm* for predictions with expert advices (see e.g. Cesa-Bianchi and Lugosi [10], Chapter 6) or Blackwell's [6] approachability theory.

We recall that in a two-person repeated game with vector payoffs in some euclidian space $\mathbb{R}^k$, a player can *approach* a given set $E \subset \mathbb{R}^k$, if he can insure that, after some stage and with a great probability, the average payoff will always remain close to E. When both players observe their opponent's moves (or at least the payoffs), Blackwell [6] proved that if E satisfies some geometrical condition – E is then called a B-set –, then Player 1 can approach it. He also deduced that either Player 1 can approach a convex set or Player 2 can exclude it, i.e. he can approach the complement of one of its neighborhood.

[*]Laboratoire de Probabilités et de Modèles Aléatoires, Université Paris 7, 175 rue du Chevaleret, 75013 Paris. vianney.perchet@normalesup.org

[†]Laboratoire de Mathématiques, UMR6205, Université de Bretagne Occidentale, 6 Avenue Le Gorgeu, 29200 Brest, France

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In the partial monitoring case, when players do not observe their opponent's moves but receive random signals (their laws may depend on the actions played), working on the space of unknown payoffs might not be sufficient – except for specific cases, such as the minimization of external regret as did Lugosi, Mannor and Stoltz [16].

Attempts were made to circumvent this issue, notably by Aumann and Maschler [4] and Kohlberg [14] in the framework of repeated games with incomplete information. Lehrer and Solan [15] also considered strategies that are defined, not as a function of the unknown past payoffs, but as a function of the past signals, and they proved the existence of strategies that satisfy an extension of the consistency property. Perchet [17] also used this approach to provide a complete characterization of approachable convex sets; it extends Blackwell's one in the full monitoring case.

Games with partial monitoring

Formally, we consider a two player repeated game Γ with partial monitoring where, at stage $n \in \mathbb{N}$, Player 1 chooses an action i_n in a finite set I and, simultaneously, Player 2 chooses $j_n \in J$. This generates a vector payoff $\rho_n = \rho(i_n, j_n) \in \mathbb{R}^k$ where ρ is a mapping from $I \times J$ to $\mathbb{R}^k$, extended to $\Delta(I) \times \Delta(J)$ by $\rho(x, y) = \mathbb{E}_{x,y}[\rho(i, j)] := \sum_{i,j} x_i y_j \rho(i, j)$, where $\Delta(I)$ and $\Delta(J)$ stand for the sets of probability measures over I and J.

The important difference with usual repeated games with full monitoring is that, at stage n, Player 1 does not observe Player 2's action j_n, nor his payoff ρ_n, but he receives a random signal $s_n \in S$ (where S is the finite set of signals) whose law is $s(i_n, j_n) \in \Delta(S)$. The mapping $s : I \times J \to \Delta(S)$, known from both players, is also extended to $\Delta(I) \times \Delta(J)$ by $s(x, y) = \mathbb{E}_{x,y}[s(i, j)] \in \Delta(S)$. On the other hand, Player 2 observes i_n, j_n and s_n.

In this framework, a strategy σ of Player 1 is a mapping from the set of past finite observations $H^1 := \bigcup_{n \in \mathbb{N}} (I \times S)^n$ into $\Delta(I)$; similarly, a strategy τ of Player 2 is a mapping from $H^2 := \bigcup_{n \in \mathbb{N}} (I \times S \times J)^n$ into $\Delta(J)$. As usual, a couple of strategies (σ, τ) generates a probability, also denoted by $\mathbb{P}_{\sigma,\tau}$ on $H^\infty := (I \times S \times J)^\infty$ endowed with the cylinder topology.

We introduce the so-called *maximal informative* mapping $\mathbf{s}$ from $\Delta(J)$ to $\Delta(S)^I$ by $\mathbf{s}(y) = (s(i, y))_{i \in I}$. Its range $\mathcal{S} \subset \Delta(S)^I$ is a polytope (i.e. the convex hull of a finite number of points) and any of its element is called a *flag*. Whatever being his move, Player 1 cannot distinguish between two actions $y_0, y_1 \in \Delta(J)$ that generate the same flag $\mu \in \mathcal{S}$, i.e. such that $\mathbf{s}(y_0) = \mathbf{s}(y_1) = \mu$, thus $\mathbf{s}(y)$ – although not observed – is the maximal information available to Player 1, given $y \in \Delta(J)$. Note that with full monitoring, a flag is simply the law of the action of Player 2.

Approachability

Given a closed set $E \subset \mathbb{R}^k$ and $\delta \geq 0$, we denote by $d(z, E) = \inf_{e \in E} \|z - e\|$ (with $\|\cdot\|$ the Euclidian norm) the distance to E, by $E^\delta = \{\omega \in \mathbb{R}^k; \ d(\omega, E) < \delta\}$ the δ-neighborhood of E and finally by $\Pi_E(z) = \{e \in E; \ d(z, E) = \|z - e\|\}$ the set of closest points to z in E (called projections of z). Given any sequence $\{a_m\}_{m \in \mathbb{N}}$ and $n \geq 1$, $\overline{a}_n = \frac{1}{n} \sum_{m=1}^n a_m$ is its average up to the n-th term.

Blackwell [6] defined approachability as follows. A closed set $E \subset \mathbb{R}^k$ is approachable by Player 1 if for every $\varepsilon > 0$, there exist a strategy σ of Player 1 and $N \in \mathbb{N}$, such that for every strategy τ of Player 2:

$$\sup_{n \geq N} \mathbb{E}_{\sigma,\tau} \left[d(\overline{\rho}_n, E) \right] \leq \varepsilon.$$

In a dual way,, a set E is excludable by Player 2, if there exists $\delta > 0$ such that the complement of E^δ is approachable by Player 2.

In words, Player 1 can approach a set $E \subset \mathbb{R}^k$ if he has a strategy such that the average payoff converges[1] to E, uniformly with respect to the strategies of Player 2.

In the case of a convex set C, Blackwell [6] and Perchet [17] (see also Kohlberg [14] for a specific case) provided a complete characterization with, respectively, full and partial monitoring. Those results can be summarized thanks to the following notation (that will furthermore ease statements of generalized results). Let X and $\mathcal{X}$ denote some *informative actions spaces* and P be a multivalued application from $X \times \mathcal{X}$ to $\mathbb{R}^k$. In our case, $X = \Delta(I)$ and $\mathcal{X} = \mathcal{S}$ and $P(x, \xi) = \left\{ \rho(x, y); \ y \in \mathbf{s}^{-1}(\xi) \right\}$ for every $x \in X$ and $\xi \in \mathcal{X}$; with full monitoring, $\mathcal{X}$ reduces to $\Delta(J)$ and $P(x, y) = \{\rho(x, y)\}$.

Both conditions of Blackwell [6] and Perchet [17] reduce to the following succinct one:

A convex set C is approachable by Player 1 if and only if $\forall \xi \in \mathcal{X}, \ \exists x \in X, \ P(x, \xi) \subset C.$
(1)

Actually, and as we shall see later, this result holds when X and $\mathcal{X}$ are any convex compact sets of some Euclidian spaces and $P : X \times \mathcal{X} \rightrightarrows \mathbb{R}^k$ is a *L-Lipschitzian convex hull*. The latter condition means that for every $x \in X$, $P(x, \cdot)$ is convex and that there exists a family $\{p_\kappa : X \times \mathcal{X} \to \mathbb{R}^k; \ \kappa \in \mathcal{K}\}$ of L-Lipschitzian functions such that, for every $(x, \xi) \in X \times \mathcal{X}$, $P(x, \xi) = \mathrm{co} \left\{ p_\kappa(x, \xi); \ \kappa \in \mathcal{K} \right\}$, where $\mathrm{co}(\cdot)$ stands for the convex hull.

Purely informative game

The introduction of two arbitrary compact sets X and $\mathcal{X}$, endowed with the weak-$\star$ topology, and a multivalued mapping P motivate the following definition of the abstract *purely informative game* $\widetilde{\Gamma}$. At stage $n \in \mathbb{N}$, Player 1 chooses $\mathbf{x}_n \in \Delta(X)$, the set of probability measures over X and, simultaneously, Player 2 chooses $\boldsymbol{\xi}_n \in \Delta(\mathcal{X})$. Those choices generate the outcome (the term payoff will only be used in Γ):

$$\theta_n = \theta(\mathbf{x}_n, \boldsymbol{\xi}_n) = \mathbf{x}_n \otimes \boldsymbol{\xi}_n \in \Delta(X \times \mathcal{X}),$$

where $\otimes$ stands for the product distribution. A strategy σ of Player 1 is now a mapping from $\bigcup_{n \in \mathbb{N}} (\Delta(X) \times \Delta(\mathcal{X}))^n$ to $\Delta(X)$ and similarly, a strategy τ of Player 2 is defined as a mapping from $\bigcup_{n \in \mathbb{N}} (\Delta(X) \times \Delta(\mathcal{X}))^n$ to $\Delta(\mathcal{X})$. With these notations, a pair of strategies (σ, τ) induces a unique sequence $(\mathbf{x}_n, \boldsymbol{\xi}_n)_{n \in \mathbb{N}}$ in $(\Delta(X) \times \Delta(\mathcal{X}))^{\mathbb{N}}$.

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[1] The almost sure convergence can also be required, but to the cost of cumbersome notations.

Let $\overline{\theta}_n$ be the average up to stage n of the measures θ_n which is defined as follows: for every Borel subset $F \subset X \times \mathcal{X}$, $\overline{\theta}_n(F) := \frac{1}{n} \sum_{m=1}^{n} \theta_m(F)$. Then a closed set $\widetilde{E} \subset \Delta(X \times \mathcal{X})$ is approachable by Player 1 if for every $\varepsilon > 0$ there exist a strategy σ of Player 1 and $N \in \mathbb{N}$ such that for every strategy τ of Player 2:

$$\forall n \geq N, W_2\left(\overline{\theta}_n, \widetilde{E}\right) := \inf_{\theta \in \widetilde{E}} W_2(\overline{\theta}_n, \theta) \leq \varepsilon,$$

where W_2 is the (quadratic) Wasserstein distance – defined later in a section devoted to preliminaries –. For the definition of approachability, any distance metrizing the weak-$\star$ convergence of measures could be suitable but for the characterization of approachability - as we will demonstrate throughout the paper - the quadratic Wasserstein distance is very convenient.

Organization and main results

The purely informative game $\widetilde{\Gamma}$ unifies the framework of both games with or without signals; we prove indeed in the first section (see Proposition 1) that a set is approachable in a game Γ if and only if its *image set* is also approachable in $\widetilde{\Gamma}$. And we exhibit in the second section, see Proposition 2, a necessary and sufficient condition under which the latter holds. So this gives immediately the same result for (non necessarily convex) approachable sets with partial monitoring, for the first time in the literature.

We investigate the case of approachable convex sets, which usually benefits from a quite simple characterization (condition (1)). We show in Section 3 that this is covered by our first main result, Theorem 3 – that is actually more general.

In the last section, we are interested in specific games (that we called convex) and thanks to a totally different notion of approachability (along with some unusual definition of convexity) we obtain, not only another characterization of approachable sets, but also rates of converges. Those main results are stated in Theorem 4 and Theorem 5.

1   Some preliminaries on Wasserstein Distance and on Normals

Here we define in a precise and concise way the distance W_2 we have already used in the introduction. We also introduce some material that will be used in the sequel. The reader can refer for this part to the books [21, 11]. Moreover the projection onto a nonconvex set in $\mathbb{R}^k$ is in general multi valued and a proper notion of normal should be used (cf for instance [2]). So we need to adapt this definition to the set of measures (following ideas of [9]).

For every μ and ν in $\Delta^2\left(\mathbb{R}^N\right)$ – the set of measures with a finite moment of order 2 in some euclidian space $\mathbb{R}^N$ –, the (squared) Wasserstein distance between μ and ν is defined by:

$$W_2^2(\mu, \nu) := \inf_{\gamma \in \Pi(\mu,\nu)} I[\gamma] = \sup_{(\phi,\psi) \in \Xi} J(\phi, \psi) = \inf_{U \sim \mu; V \sim \nu} \mathbb{E}\left[\|U - V\|^2\right], \text{ where} \quad (2)$$

– $U \sim \mu$ means that the law of the random variable U is μ;

– $\Pi(\mu, \nu)$ is the set of probability measures $\gamma \in \Delta\left(\mathbb{R}^N \times \mathbb{R}^N\right)$ with first marginal μ and second marginal ν and

$$I[\gamma] = \int_{\mathbb{R}^N \times \mathbb{R}^N} \|x - y\|^2 \mathrm{d}\gamma(x, y) \,;$$

– Ξ is the set of functions $(\phi, \psi) \in L^1_\mu(\mathbb{R}^N, \mathbb{R}) \times L^1_\nu(\mathbb{R}^N, \mathbb{R})$ such that $\phi(x) + \psi(y) \leq \|x - y\|^2$, $\mu \otimes \nu$-as and

$$J(\phi, \psi) = \int_{\mathbb{R}^N} \phi \mathrm{d}\mu + \int_{\mathbb{R}^N} \psi \mathrm{d}\nu.$$

Furthermore, if the supports of μ and ν are included in a compact set, then we can assume that Ξ is reduced to the set of functions (ϕ, ϕ^*) such that, for some arbitrarily fixed $x_0 \in K$,

$$\phi^*(x) = \inf_{y \in K} \|x - y\|^2 - \phi(y), \quad \phi = (\phi^*)^* \text{ and } \phi(x_0) = 0.$$

Since every function in Ξ is $2\|K\|$-lipschitzian (where $\|K\|$ is the diameter of K), Arzela-Ascoli's theorem implies that $(\Xi, \|\|_\infty)$ is relatively compact.

Actually, the infimum and supremum in (2) are achieved; we denote by $\Phi(\mu, \nu)$ the subset of Ξ that maximizes $J(\phi, \phi^*)$ and its elements are called *Kantorovitch potentials* from μ to ν. Any probability measure $\gamma \in \Delta\left(\mathbb{R}^{2N}\right)$ that achieves the minimum is an *optimal plan* from μ to ν.

More details on the definition of W_2, based on Kantorovitch duality, can be found for example in Dudley [11], chapter 11.8 or Villani [21], chapter 2.

Brenier's [8] theorem states that if $\mu \ll_0 \boldsymbol{\lambda}$ (i.e., the probability measure μ is absolutely continuous with respect to the Lebesgue measure $\boldsymbol{\lambda}$ and has a strictly positive density), then there exist a unique optimal plan γ and a unique convex Kantorovitch potential from μ to ν. They satisfy:

$$d\gamma(x, y) = d\mu(x)\delta_{\{x - \nabla\phi(x)\}} \text{ or } \text{ equivalently } \gamma = (\mathrm{Id} \times (\mathrm{Id} - \nabla\phi)) \,\sharp \mu,$$

where for any $\psi : \mathbb{R}^N \to \mathbb{R}^N$, Borel measurable with at most a linear growth, $\psi\sharp\mu \in \Delta\left(\mathbb{R}^N\right)$ is the push-forward of μ by ψ – also called the image probability measure of μ by ψ. It is defined by

$$\psi\sharp\mu(A) = \mu\left(\psi^{-1}(A)\right) \qquad \forall A \subset \mathbb{R}^N, \text{ Borel measurable}$$

or equivalently by: for every Borel measurable bounded maps $f : \mathbb{R}^N \to \mathbb{R}$:

$$\int_{\mathbb{R}^N} f \mathrm{d}(\psi\sharp\mu) = \int_{\mathbb{R}^N} f(\psi(x)) \mathrm{d}\mu(x).$$

A classical approximation result (see e.g. Dudley [11])) is that for any convex compact $K \subset \mathbb{R}^N$ with non-empty interior and any $\varepsilon > 0$, there exists $\Delta_\varepsilon(K)$ a compact subset of $\Delta_0(K) = \{\mu \in \Delta(K), \mu \ll_0 \boldsymbol{\lambda}\}$ such that, for every $\mu \in \Delta(K)$, $W_2^2(\mu, \Delta_\varepsilon(K)) \leq \varepsilon$. So Brenier's theorem actually implies that

$$\Phi \text{ is singlevalued and uniformly continuous on } \Delta_\varepsilon(K) \times K.$$

Some geometrical properties of W_2

Blackwell's approachability results rely deeply on the geometry of Euclidian spaces. One of our goals is to underline and establish required results for the measure space equipped with W_2. For instance, in Euclidian (and also Hilbert) spaces the projection onto a closed convex set could be characterized equivalently by the minimization of the distance to the set or by a characterization of the projection by the well-known condition with scalar products. The Lemma 6 could be viewed as a way of writing this "condition with scalar products" in the space of measures. Also Blackwell's conditions requires suitable notions of projections and normals we will introduce now.

In the space $\Delta(K)$ equipped with W_2 we define the corresponding definitions of proximal normals (see Bony [7]) to nonempty closed sets $A \subset \Delta(K)$ at some $\underline{\mu} \in A$. As usual, we say that $\underline{\mu}$ is a projection (with respect to the Wasserstein distance) of a measure $\mu \in \Delta(K)$ if $\overline{W}_2(\mu, A) := \inf_{\theta \in A} W_2(\mu, \theta) = W_2(\mu, \underline{\mu})$.

Actually, proximal normals can be defined in two different ways, depending on which equivalent definition of W_2 is used.

– *Proximal potential normal*: a continuous function $\phi : K \to \mathbb{R}$ is a proximal potential normal to A at $\underline{\mu}$ if ϕ is a Kantorovitch potential from $\underline{\mu}$ to some $\mu \in \Delta(K)$ where $\underline{\mu}$ is a projection of μ on A.

$$\mathrm{NC}_A^p\left(\underline{\mu}\right) = \left\{\phi : K \to \mathbb{R}; \ \phi \text{ proximal potential normals to } A \text{ at } \underline{\mu}\right\}.$$

– *Proximal gradient normal* (adapted from [1]): a map $p \in L_\mu^2(\mathbb{R}^N, \mathbb{R}^N)$ is a proximal gradient normal to A at $\underline{\mu}$ if $p \in \mathcal{P}^\gamma(\underline{\mu}, \mu)$ for some $\mu \notin A$ that projects on $\underline{\mu}$ and some optimal plan $\gamma \in \Pi(\underline{\mu}, \mu)$, where, for every $\mu, \nu \in \Delta(\mathbb{R}^N)$ and $\gamma \in \Pi(\mu, \nu)$,

$$\mathcal{P}^\gamma(\underline{\mu}, \mu) = \left\{p \in L_\mu^2\left(\mathbb{R}^N, \mathbb{R}^N\right) ; \ \forall \psi \in \mathcal{C}_{LB}, \int_{\mathbb{R}^N} \langle \psi(x), p(x)\rangle \mathrm{d}\mu(x) = \int_{\mathbb{R}^{2N}} \langle \psi(x), x - y\rangle \mathrm{d}\gamma(x, y)\right\}$$

with $\mathcal{C}_L B$ the sets of Borel measurable map $\psi : \mathbb{R}^N \to \mathbb{R}^N$ with at most a linear growth. Riesz representation theorem ensures the non-emptiness of $\mathcal{P}^\gamma(\underline{\mu}, \mu)$ (see [9]).

$$\mathrm{NC}_A^g\left(\underline{\mu}\right) = \left\{p \in L_\mu^2(\mathbb{R}^N, \mathbb{R}^N); \ p \text{ proximal gradient normals to } A \text{ at } \underline{\mu}\right\}.$$

Observe that Brenier's Theorem also implies that both definitions of proximal normals are, in some sense, quite close. Indeed, if A is a compact subset of $\Delta(K)$ and $\underline{\mu} \in A$ and $\mu \ll_0 \boldsymbol{\lambda}$ then

$$\phi \in \mathrm{NC}_A^p(\underline{\mu}) \Longrightarrow \nabla \phi \in \mathrm{NC}_A^g(\underline{\mu}).$$

2 Equivalences between approachability in both games

Recall that we represent a game Γ by two compact convex action spaces X and $\mathcal{X}$ and a L-Lipschitzian convex hull P. We define the set of outcomes compatible with

$\theta \in \Delta(\mathcal{X} \times \mathcal{X})$ by:

$$\rho(\theta) = \int_{\mathcal{X} \times \mathcal{X}} P(x, \xi) \mathrm{d}\theta \subset \mathbb{R}^k$$

where the integral is in Aumann [3]'s sense: it is the set of all integrals of measurable selection of P. For every subset $\widetilde{E} \subset \Delta(\mathcal{X} \times \mathcal{X})$, the set of compatible outcomes $\rho(\widetilde{E}) \subset \mathbb{R}^k$ is defined by:

$$\rho(\widetilde{E}) = \left\{ \rho(\theta);\ \theta \in \widetilde{E} \right\} \subset \mathbb{R}^k.$$

Reciprocally, for every $E \subset \mathbb{R}^k$, the set of compatible measures $\rho^{-1}(E) \subset \Delta(\mathcal{X} \times \mathcal{X})$ is defined by:

$$\rho^{-1}(E) = \{ \theta \in \Delta(\mathcal{X} \times \mathcal{X});\ \rho(\theta) \subset E \}.$$

The mapping $\mathbf{s}$ does not appear in the description of $\widetilde{\Gamma}$ but only in the definition of ρ that links Γ and $\widetilde{\Gamma}$: given a set $E \subset \mathbb{R}^k$ in Γ, we introduced the image set $\widetilde{E} = \rho^{-1}(E) \subset \Delta(\mathcal{X} \times \mathcal{X})$ in $\widetilde{\Gamma}$. And it is quite intuitive that E is approachable if and only $\widetilde{E}$ is (see Proposition 1 below, whose proof – mainly technical – is delayed to the Appendix in order to keep some fluency).

Proposition 1

    i) A set $E \subset \mathbb{R}^k$ is approachable in Γ if and only if $\rho^{-1}(E) \subset \Delta(\mathcal{X} \times \mathcal{X})$ is approachable in $\widetilde{\Gamma}$;

    ii) If a set $\widetilde{E} \subset \Delta(\mathcal{X} \times \mathcal{X})$ is approachable in $\widetilde{\Gamma}$ then the set $\rho(\widetilde{E}) \subset \mathbb{R}^k$ is approachable in Γ;

    iii) For every convex set $C \subset \mathbb{R}^k$, $\rho^{-1}(C) \subset \Delta(\mathcal{X} \times \mathcal{X})$ is a (possibly empty) convex set and for every convex set $\widetilde{C} \subset \Delta(\mathcal{X} \times \mathcal{X})$, $\rho(\widetilde{C})$ is a convex set.

Notice that point *ii)* cannot be an equivalence. Consider the case $\rho = 0$ and $\widetilde{E} = \{\mathbf{x_0} \otimes \mathbf{y_0}\}$ for some $\mathbf{x_0}$ and $\mathbf{y_0}$. Then $\rho(\widetilde{E}) = \{0\}$ is approachable but $\widetilde{E}$ is not, since Player 2 just has to play $\mathbf{y_1} \neq \mathbf{y_0}$ at each stage. This is a consequence of the usual inclusions $\rho\left(\rho^{-1}(E)\right) \subset E$ and $\rho^{-1}\left(\rho(\widetilde{E})\right) \supset \widetilde{E}$.

3   Approachability of $\widetilde{B}$-sets

Blackwell [6] noticed that a closed set E that fulfills the following geometrical condition – E is then call a B-set – is approachable by Player 1 with full monitoring. Formally, a closed subset E of $\mathbb{R}^k$ is a B-set, if

$$\forall z \in \mathbb{R}^k,\ \exists p \in \Pi_E(z),\ \exists x \in \Delta(I),\ \langle \rho(x, y) - p, z - p \rangle \leq 0,\quad \forall y \in \Delta(J).$$

An equivalent formulation using $\mathrm{NC}_E(q)$, the set of proximal normals to E at q, appeared in [2]. Indeed E is a B-set if and only if

$$\forall p \in E,\ \forall q \in \mathrm{NC}_E(p),\ \exists x \in \Delta(I),\ \forall y \in \Delta(J),\ \langle \rho(x, y) - p, q \rangle \leq 0.$$

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Blackwell [6] and Spinat [20] proved that E is approachable in Γ^f if and only if it contains a B-set.

Our definition of proximal potential normals gives to $\mathbf{W}_2$ a structure close to a Hilbert. This allows to extend Blackwell's definition of a B-set as follows.

Definition 1 *A set $\widetilde{E} \subset \Delta(X \times \mathcal{X})$ is a $\widetilde{B}$-set if for every θ not in $\widetilde{E}$ there exist $\underline{\theta} \in \Pi_{\widetilde{E}}(\theta)$, $\phi \in \Phi(\underline{\theta}, \theta)$ and $\mathbf{x}\, (= \mathbf{x}(\theta)) \in \Delta(X)$ such that :*

$$\int_{X \times \mathcal{X}} \phi\; \mathrm{d}(\underline{\theta} - \mathbf{x} \otimes \boldsymbol{\xi}) \le 0, \quad \forall \boldsymbol{\xi} \in \Delta(\mathcal{X}).$$

Or stated in terms of proximal potential normals:

$$\forall \underline{\theta} \in \widetilde{E},\, \forall \phi \in \mathrm{NC}^p_{\widetilde{E}}(\underline{\theta}),\, \exists \mathbf{x} \in \Delta(X),\, \forall \boldsymbol{\xi} \in \Delta(\mathcal{X}),\, \int_{X \times \mathcal{X}} \phi\; \mathrm{d}(\underline{\theta} - \mathbf{x} \otimes \boldsymbol{\xi}) \le 0.$$

The concept of $\widetilde{B}$-sets is indeed the natural extension of B-sets because of the following proposition.

Proposition 2 *A set $\widetilde{E} \subset \Delta(X \times \mathcal{X})$ is approachable if and only if it contains a $\widetilde{B}$-set.*

Proof: We only prove here the sufficient part, i.e. a $\widetilde{B}$-set is approachable by Player 1 (by adapting Blackwell [6]'s ideas to our framework). Again, we postpone the proof of the necessary part (almost identical to the full monitoring case) to the Appendix to prevent cumbersomeness.

Let $\varepsilon > 0$ be fixed. For every probability distribution $\theta \in \Delta(X \times \mathcal{X})$, we denote by $\theta^\varepsilon \in \Delta_\varepsilon(X \times \mathcal{X})$ any arbitrary approximation of θ such that $W_2^2(\theta, \theta^\varepsilon) \le \varepsilon$.

Consider the strategy σ^ε of Player 1 that plays, at stage $n \in \mathbb{N}$, $\mathbf{x}(\bar{\theta}^\varepsilon_{n-1})$ given by the definition of a $\widetilde{B}$-set, where $\bar{\theta}^\varepsilon_{n-1}$ is the average of the $n-1$ first $(\theta_m)^\varepsilon$. Then, if we denote by $\underline{\theta}^\varepsilon_{n-1}$ the projection of $\bar{\theta}^\varepsilon_{n-1}$ over $\widetilde{E}$ and let $w_n = W_2^2\left(\bar{\theta}^\varepsilon_n, \widetilde{E}\right)$:

$$w_n = W_2^2\left(\bar{\theta}^\varepsilon_n, \widetilde{E}\right) \le W_2^2\left(\bar{\theta}^\varepsilon_n, \underline{\theta}^\varepsilon_{n-1}\right) = W_2\left(\underline{\theta}^\varepsilon_{n-1}, \frac{n-1}{n}\bar{\theta}^\varepsilon_{n-1} + \frac{\theta^\varepsilon_n}{n}\right)$$

$$= \sup_{\phi \in \Xi} \int_{X \times \mathcal{X}} \phi\; \mathrm{d}\underline{\theta}^\varepsilon_{n-1} + \int_{X \times \mathcal{X}} \phi^* \mathrm{d}\left(\frac{n-1}{n}\bar{\theta}^\varepsilon_{n-1} + \frac{\theta^\varepsilon_n}{n}\right)$$

$$= \int_{X \times \mathcal{X}} \phi_n\; \mathrm{d}\underline{\theta}^\varepsilon_{n-1} + \int_{X \times \mathcal{X}} \phi_n^* \mathrm{d}\left(\frac{n-1}{n}\bar{\theta}^\varepsilon_{n-1} + \frac{\theta^\varepsilon_n}{n}\right)$$

$$\le \frac{n-1}{n} w_{n-1} + \frac{1}{n}\left(\int_{X \times \mathcal{X}} \phi_n \mathrm{d}\underline{\theta}^\varepsilon_{n-1} - \int_{X \times \mathcal{X}} \phi_n^* \mathrm{d}\theta^\varepsilon_n\right)$$

where ϕ_n is the optimal Kantorovitch potential from $\underline{\theta}^\varepsilon_{n-1}$ to $\frac{n-1}{n}\bar{\theta}^\varepsilon_{n-1} + \frac{\theta^\varepsilon_n}{n}$. Let us denote by ϕ_0 the optimal Kantorovitch potential from $\underline{\theta}^\varepsilon_{n-1}$ to $\bar{\theta}^\varepsilon_{n-1}$ and by $\omega^\varepsilon(\cdot)$ the modulus of continuity of Φ restricted to the compact set $\widetilde{E} \times \Delta_\varepsilon(X \times \mathcal{X})$.

The definition of W_2^2 implies that

$$W_2^2\left(\bar{\theta}^\varepsilon_{n-1}, \frac{n-1}{n}\bar{\theta}^\varepsilon_{n-1} + \frac{\theta^\varepsilon_n}{n}\right) \le \frac{1}{n} W_2^2(\bar{\theta}^\varepsilon_{n-1}, \theta^\varepsilon_n) \le \frac{4\|X \times \mathcal{X}\|^2}{n},$$

therefore $\|\phi_0 - \phi_n\|_\infty \leq \omega^\varepsilon(2\|X\|/\sqrt{n})$ and

$$w_n \leq \frac{n-1}{n}w_{n-1} + \frac{1}{n}\left(\int_{X \times \mathcal{X}} \phi_0 \mathrm{d}\underline{\theta}_{n-1}^\varepsilon + \int_{X \times \mathcal{X}} \phi_0^* \mathrm{d}\theta_n^\varepsilon\right) + \frac{2}{n}\omega\left(\frac{2\|X \times \mathcal{X}\|}{\sqrt{n}}\right).$$

Recall that θ_n^ε is such that that $\int_{X \times \mathcal{X}} \phi_0 \mathrm{d}\theta_n + \int_{X \times \mathcal{X}} \phi_0^* \mathrm{d}\theta_n^\varepsilon \leq W_2^2(\theta_n, \theta_n^\varepsilon,) \leq \varepsilon$, therefore

$$w_n \leq \frac{n-1}{n}w_{n-1} + \frac{1}{n}\left(\int_{X \times \mathcal{X}} \phi_0 \mathrm{d}\underline{\theta}_{n-1}^\varepsilon - \int_{X \times \mathcal{X}} \phi_0 \mathrm{d}\theta_n\right) + \frac{2}{n}\omega\left(\frac{2\|X \times \mathcal{X}\|}{\sqrt{n}}\right) + \frac{\varepsilon}{n}.$$

Since $\widetilde{E}$ is a $\widetilde{B}$-set and because of the choice of $\mathbf{x}_n \in \Delta(X)$, for every $\boldsymbol{\xi}_n \in \Delta(\mathcal{X})$, $\int_{X \times \mathcal{X}} \phi_0 \mathrm{d}\left(\underline{\theta}_{n-1}^\varepsilon - \mathbf{x}_n \otimes \boldsymbol{\xi}_n\right) \leq 0$, thus

$$w_n \leq \frac{n-1}{n}w_{n-1} + \frac{2}{n}\omega^\varepsilon\left(\frac{2\|X \times \mathcal{X}\|}{\sqrt{n}}\right) + \frac{\varepsilon}{n}$$

and this yields, by induction, that

$$W_2^2\left(\bar{\theta}_n^\varepsilon, \widetilde{E}\right) \leq \frac{1}{n+1}W_2^2\left(\overline{\theta^\varepsilon}_1, \widetilde{E}\right) + \frac{2}{n+1}\sum_{m=1}^n \omega^\varepsilon\left(\frac{2\|X \times \mathcal{X}\|}{\sqrt{k}}\right) + \varepsilon.$$

Since $\omega^\varepsilon(2\|X \times \mathcal{X}\|/\sqrt{k})$ converges to 0 when k goes to infinity, then w_n is asymptotically smaller than ε. The fact that $W_2\left(\bar{\theta}_n, \widetilde{E}\right) \leq W_2\left(\bar{\theta}_n^\varepsilon, \widetilde{E}\right) + \varepsilon$ implies that $\widetilde{E}$ is approachable by Player 1. $\square$

4 Characterization of convex approachable sets

There also exists in $\widetilde{\Gamma}$ a complete characterization of approachable convex sets :

Theorem 3 *A convex set $\widetilde{C}$ is approachable if and only if:*

$$\forall \boldsymbol{\xi} \in \Delta(\mathcal{X}), \exists \mathbf{x} \in \Delta(X), \ \theta(\mathbf{x}, \boldsymbol{\xi}) = \mathbf{x} \otimes \boldsymbol{\xi} \in \widetilde{C}.$$

Proof: Once again, we will follow the idea of Blackwell. Assume that there exists $\boldsymbol{\xi}$ such that, for every $\mathbf{x} \in \Delta(X)$, $\theta(\mathbf{x}, \boldsymbol{\xi}) \notin \widetilde{C}$. The application $\mathbf{x} \mapsto W_2\left(\theta(\mathbf{x}, \boldsymbol{\xi}), \widetilde{C}\right)$ is continuous on the compact set $\Delta(X)$, therefore there exists $\delta > 0$ such that $W_2^2\left(\theta(\mathbf{x}, \boldsymbol{\xi}), \widetilde{C}\right) \geq \delta$.

Consider the strategy of Player 2 that consists of playing $\boldsymbol{\xi}$ at every stages, then $\theta_n = \mathbf{x}_n \otimes \boldsymbol{\xi}$, $\bar{\theta}_n = \bar{\mathbf{x}}_n \otimes \boldsymbol{\xi} = \theta(\bar{\mathbf{x}}_n, \boldsymbol{\xi})$ and $W_2^2\left(\bar{\theta}_n, \widetilde{C}\right) \geq \delta > 0$. Therefore $\widetilde{C}$ is not approachable by Player 1.

Reciprocally, assume that for every $\boldsymbol{\xi} \in \Delta(\mathcal{X})$ there exists $\mathbf{x} \in \Delta(X)$ such that $\theta(\mathbf{x}, \boldsymbol{\xi}) \in \widetilde{C}$. We claim that this implies that $\widetilde{C}$ is a $\widetilde{B}$-set.

Let $\overline{\theta}$ be a probability measure that does not belong to $\widetilde{C}$ and assume (for the moment) that $\overline{\theta} \ll_0 \boldsymbol{\lambda}$. Denote by $\underline{\theta} \in \widetilde{C}$ any of its projection then, by definition of the projection and convexity of $\widetilde{C}$:

$$
\begin{aligned}
W_2(\underline{\theta}, \overline{\theta}) \le W_2^2\left((1-\lambda)\underline{\theta} + \lambda \mathbf{x} \otimes \boldsymbol{\xi}, \overline{\theta}\right) &= \sup_{\phi \in \Xi} \int_{X \times \mathcal{X}} \phi \mathrm{d}\left((1-\lambda)\underline{\theta} + \lambda \mathbf{x} \otimes \boldsymbol{\xi}\right) + \int_{X \times \mathcal{X}} \phi^* \mathrm{d}\overline{\theta} \\
&= \sup_{\phi \in \Xi} \int_{X \times \mathcal{X}} \phi \mathrm{d}\underline{\theta} + \int_{X \times \mathcal{X}} \phi^* \mathrm{d}\overline{\theta} - \lambda \int_{X \times \mathcal{X}} \phi \mathrm{d}(\underline{\theta} - \mathbf{x} \otimes \boldsymbol{\xi}) \\
&= \int_{X \times \mathcal{X}} \phi_\lambda \mathrm{d}\underline{\theta} + \int_{X \times \mathcal{X}} \phi_\lambda^* \mathrm{d}\overline{\theta} - \lambda \int_{X \times \mathcal{X}} \phi_\lambda \mathrm{d}(\underline{\theta} - \mathbf{x} \otimes \boldsymbol{\xi}) \\
&\le \int_{X \times \mathcal{X}} \phi_0 \mathrm{d}\underline{\theta} + \int_{X \times \mathcal{X}} \phi_0^* \mathrm{d}\overline{\theta} - \lambda \int_{X \times \mathcal{X}} \phi_\lambda \mathrm{d}(\underline{\theta} - \mathbf{x} \otimes \boldsymbol{\xi}) \\
&= W_2^2(\overline{\theta}, \underline{\theta}) - \lambda \int_{X \times \mathcal{X}} \phi_\lambda \mathrm{d}(\underline{\theta} - \mathbf{x} \otimes \boldsymbol{\xi})
\end{aligned}
$$

where ϕ_λ (resp. ϕ_0) is the unique potential from $(1-\lambda)\underline{\theta} + \lambda \mathbf{x} \otimes \boldsymbol{\xi}$ (resp. $\underline{\theta}$) to $\overline{\theta}$. Therefore, for every $\lambda > 0$, $\lambda \int_{X \times \mathcal{X}} \phi_\lambda \mathrm{d}(\underline{\theta} - \mathbf{x} \otimes \boldsymbol{\xi}) \le 0$. Dividing by $\lambda > 0$ yields:

$$
\int_{X \times \mathcal{X}} \phi_\lambda \mathrm{d}(\underline{\theta} - \mathbf{x} \otimes \boldsymbol{\xi}) \le 0, \quad \forall \lambda > 0.
$$

Since $(1-\lambda)\underline{\theta} + \lambda \mathbf{x} \otimes \boldsymbol{\xi}$ converges to $\underline{\theta}$, any accumulation point of $(\phi_\lambda)_{\lambda > 0}$ has to belong (for every $\mathbf{x}$ and $\boldsymbol{\xi}$) to $\Phi(\underline{\theta}, \overline{\theta}) = \{\phi_0\}$. Stated differently, given $\phi_0 \in \Phi(\underline{\theta}, \overline{\theta})$, one has:

$$
\max_{\boldsymbol{\xi} \in \Delta(\mathcal{X})} \min_{\mathbf{x} \in \Delta(X)} g_{\phi_0}(\mathbf{x}, \boldsymbol{\xi}) := \max_{\boldsymbol{\xi} \in \Delta(\mathcal{X})} \min_{\mathbf{x} \in \Delta(X)} \int_{X \times \mathcal{X}} \phi_0 \mathrm{d}(\underline{\theta} - \mathbf{x} \otimes \boldsymbol{\xi}) \le 0.
$$

The function g_{ϕ_0} is linear in both of its variable, so Sion's Theorem implies that

$$
\max_{\boldsymbol{\xi} \in \Delta(\mathcal{X})} \min_{\mathbf{x} \in \Delta(X)} g_{\phi_0}(\mathbf{x}, \boldsymbol{\xi}) = \min_{\mathbf{x} \in \Delta(X)} \max_{\boldsymbol{\xi} \in \Delta(\mathcal{X})} g_{\phi_0}(\mathbf{x}, \boldsymbol{\xi})
$$

hence $\widetilde{C}$ is a $\widetilde{B}$-set.

Assume now that $\overline{\theta} \notin \Delta_0(X \times \mathcal{X})$ and let $\overline{\theta}_n \in \Delta_{\frac{1}{n}}(X \times \mathcal{X})$ be a sequence of measures that converges to $\overline{\theta}$, $(\underline{\theta}_n)_{n \in \mathbb{N}}$ a sequence of their projections, and $\phi_n \in \Phi(\underline{\theta}_n, \overline{\theta}_n)$. Up to subsequences, we can assume that $\underline{\theta}_n$ and ϕ_n converge respectively to $\underline{\theta}$ and ϕ_0. Necessarily, $\underline{\theta}$ is a projection of $\overline{\theta}$ onto $\widetilde{C}$ and ϕ_0 belongs to $\Phi(\underline{\theta}, \overline{\theta})$. Therefore:

$$
\int_{X \times \mathcal{X}} \phi_0 \mathrm{d}(\underline{\theta} - \mathbf{x} \otimes \boldsymbol{\xi}) = \lim_{n \to \infty} \int_{X \times \mathcal{X}} \phi_n \mathrm{d}(\underline{\theta} - \mathbf{x} \otimes \boldsymbol{\xi}) \le 0
$$

and $\widetilde{C}$ is a $\widetilde{B}$-set. $\qquad \square$

Let us go back and quickly show that the characterization (1) of approachable convex sets with full monitoring is a consequence of Theorem 3:

Proof of characterization (1) By Proposition 1, a convex subset C of $\mathbb{R}^k$ is approachable if and only if the convex set $\rho^{-1}(C) \subset \Delta(X \times \mathcal{X})$ is approachable in $\widetilde{\Gamma}$. Therefore, using Theorem 3, if and only if for every $\boldsymbol{\xi} \in \Delta(\mathcal{X})$, there exists $\mathbf{x} \in \Delta(X)$ such that $\rho(\mathbf{x} \otimes \boldsymbol{\xi}) \subset C$. Let us denote by $\mathbb{E}_{\mathbf{x}} \in X$ and $\mathbb{E}_{\boldsymbol{\xi}} \in \mathcal{X}$ the expectations of any $\mathbf{x} \in \Delta(X)$ and $\boldsymbol{\xi} \in \Delta(\mathcal{X})$. Then, in the case of games with partial monitoring, one has that $\rho(\delta_{\mathbb{E}_{\mathbf{x}}} \otimes \delta_\xi) \subset \rho(\mathbf{x} \otimes \delta_\xi)$ and $\rho(\delta_x \otimes \boldsymbol{\xi}) \subset \rho(\delta_x \otimes \delta_{\mathbb{E}_{\boldsymbol{\xi}}})$.

Assume that C is approachable; since the condition holds in particular for $\boldsymbol{\xi} = \delta_\xi$, there exists $\mathbf{x}$ such that $\rho(\mathbf{x} \otimes \delta_\xi) \subset C$, therefore some $x \in X$ such that $\rho(\delta_x \otimes \delta_\xi) \subset C$ (one just has to take $x = \mathbb{E}_{\mathbf{x}}$).

Reciprocally, if C is not approachable; there exists $\boldsymbol{\xi}$ such that (in particular) for any δ_x, $\rho(\boldsymbol{\xi} \otimes \delta_x) \not\subset C$, therefore there exists some $\xi \in \mathcal{X}$ such that $\rho(\delta_\xi \otimes \delta_x) \not\subset C$ (one just has to take $\xi = \mathbb{E}_{\boldsymbol{\xi}}$).

We obtain the stated result as a consequence:

C is approachable if and only if $\forall\, \xi \in \mathcal{X},\ \exists\, x \in X,\ \rho(\delta_x \otimes \delta_\xi) = P(x, \xi) \subset C$

$\square$

This also explains why there exist convex sets that are neither approachable, nor excludable with partial monitoring (see Perchet [17]) which cannot occur with full monitoring (see Blackwell [6]): it simply due to the fact that $\rho^{-1}(C)$ can be empty.

5 Convex games

We restrict ourselves in this section to the particular class of games called *convex games* which have the following property: for every $q \in \Delta(X \times \mathcal{X})$:

$$\rho(q) = \int_{X \times \mathcal{X}} P(x, \xi) \mathrm{d}q(x, \xi) \subset P\left(\mathbb{E}_q\left[x\right], \mathbb{E}_q\left[\xi\right]\right).$$

For instance, this reduces in games with full monitoring to $\rho(q) = \rho\left(\mathbb{E}_q[x], \mathbb{E}_q[\xi]\right)$.

Example 1 *The following game where the payoffs of Player 1 are given by the matrix on the left and signals by the matrix on the right, is convex.*

	L	C	R			L	C	R
T	(0,-1)	(1,-2)	(2,-4)		T	a	a	b
B	(1,0)	(2,-1)	(3,-3)		B	a	a	b

In this game $I = \{T, B\}$, $J = \{L, C, R\}$ and $S = \{a, b\}$. If Player 1 receives the signal a, he does not know whether Player 2 used the action L or C.

We introduce the notions of displacement interpolation and convexity (see e.g. Villani [21] for more details) that will play the role of classic linear interpolation and convexity.

Given μ, $\nu \in \Delta^2(\mathbb{R}^N)$ and $t \in [0, 1]$, a displacement interpolation between μ and ν at time t is defined by $\widehat{\mu}_t = \sigma_t \sharp \gamma$, where $\gamma \in \Pi(\mu, \nu)$ is an optimal plan and σ_t is the mapping defined by $\sigma_t(x, y) = (1 - t)x + ty$. A set $\widehat{C}$ is displacement convex if for every $\mu, \nu \in \widehat{C}$, every $t \in [0, 1]$ and every optimal plan $\gamma \in \Pi(\mu, \nu)$, $\sigma_t \sharp \gamma \in \widehat{C}$.

Let $\widehat{\Gamma}$ be a new game defined as follows. At stage $n \in \mathbb{N}$, Player 1 (resp. Player 2) chooses $x_n \in X$ (resp. $y_n \in \mathcal{X}$) and the payoff is $\theta_n = \delta_{x_n} \otimes \delta_{y_n} = \delta_{(x_n, y_n)} \in \Delta(X \times \mathcal{X})$. We do not consider average payoffs in the usual sense (as in $\widetilde{\Gamma}$) but we define a sequence of recursive interpolation by:

$$\widehat{\theta}_{n+1} = \sigma_{\frac{1}{n+1}} \sharp \gamma_{n+1}, \text{ where } \gamma_{n+1} \in \Pi(\widehat{\theta}_n, \theta_{n+1}) \text{ is an optimal plan.}$$

By induction, this implies that $\widehat{\theta}_n = \delta_{\overline{x}_n} \otimes \delta_{\overline{y}_n}$. Indeed, $\widehat{\theta}_1 = \delta_{x_1} \otimes \delta_{y_1}$ and $\theta_2 = \delta_{x_2} \otimes \delta_{y_2}$ therefore:

$$\gamma_2 = (\delta_{x_1} \otimes \delta_{y_1}) \otimes (\delta_{x_1} \otimes \delta_{y_1}) \text{ and } \widehat{\theta}_2 = \sigma_{\frac{1}{2}} \sharp \gamma_2 = \delta_{\frac{x_1 + x_2}{2}} \otimes \delta_{\frac{y_1 + y_2}{2}}.$$

Definition 2 *A closed set $\widehat{E} \subset \Delta(X \times \mathcal{X})$ is displacement approachable by Player 1 if for every $\varepsilon > 0$ there exist a strategy σ of Player 1 and $N \in \mathbb{N}$ such that for every strategy τ of Player 2:*

$$\forall n \geq N, W_2\left(\widehat{\theta}_n, \widehat{E}\right) \leq \varepsilon.$$

Consider any set $E \subset \mathbb{R}^d$ and assume that $\rho^{-1}(E)$ is displacement approachable by Player 1. Since $\widehat{\theta}_n = \delta_{\overline{x}_n} \otimes \delta_{\overline{y}_n}$, the convexity of the game implies that $\rho(\overline{\theta}_n) \subset \rho(\widehat{\theta}_n)$ and thus $\rho^{-1}(E)$ is also approachable in the sense of $\widetilde{\Gamma}$. The use of displacement approachability provides however explicit and optimal bounds (see Theorem 4 below). This is the reason we investigate this special case.

In this framework, we use proximal gradient normals to define a $\widehat{B}$-set:

Definition 3 *A closed subset $\widehat{E} \subset \Delta(X \times \mathcal{X})$ is a $\widehat{B}$-set if for every θ not in $\widehat{E}$ there exist a projection $\underline{\theta} \in \Pi_{\widehat{E}}(\mu)$, $\overline{p} \in NP_{\widehat{E}}^g(\underline{\theta})$ and $x = x(\theta) \in \Delta(X)$ such that for every $y \in \mathcal{X}$, there exists an optimal plan $\gamma(x, y) \in \Pi(\underline{\theta}, \delta_x \otimes \delta_y)$ and $p(x, y) \in \mathcal{P}(\gamma(x, y))$ such that:*

$$\langle \overline{p}, p(x, y) \rangle_{L_2(\underline{\theta})} \leq 0.$$

This notion of $\widehat{B}$-set extends Blackwell's one to $\widehat{\Gamma}$ because of the following Theorems 4 and 5.

Theorem 4 *A set $\widehat{E}$ is approachable in $\widehat{\Gamma}$ if and only if it contains a $\widehat{B}$-set. Given a $\widehat{B}$-set, the strategy described by $x_{n+1} = x(\widehat{\theta}_n)$ ensures that $W_2\left(\widehat{\theta}_n, \widehat{E}\right) \leq K/\sqrt{n}$, for some $K > 0$*

Proof: Assume that Player 1 plays, at stage n, $x_n = x(\widehat{\theta}_{n-1})$ and denote by $\theta_n = \delta_{x_n} \otimes \delta_{y_n}$ the outcome at stage n. For every $n \in \mathbb{N}$, the displacement average outcome is $\widehat{\theta}_n = \delta_{\overline{x}_n} \otimes \delta_{\overline{y}_n} = \delta_{(\overline{x}_n, \overline{y}_n)}$.

If we denote by $\underline{\theta}_n \in \widehat{E}$ the projection of $\widehat{\theta}_n$ on $\widehat{E}$, then the optimal plan from $\underline{\theta}_n$ to $\widehat{\theta}_n$ is $\underline{\theta}_n \otimes \widehat{\theta}_n$. So the proximal normal $\overline{p}_n \in NP_{\widehat{E}}(\underline{\theta}_n)$ is defined by $\overline{p}_n(z) = z - (\overline{x}_n, \overline{y}_n)$. Similarly, $\underline{\theta}_n \otimes (\delta_{x_{n+1}, y_{n+1}})$ is an optimal plan from $\underline{\theta}_n$ to $\delta_{x_{n+1}} \otimes \delta_{y_{n+1}}$, thus if we define $p_{n+1}(z) = z - (x_{n+1}, y_{n+1})$, the assumption that $\widehat{E}$ is a $\widehat{B}$-set (along with the choice of $x_{n+1} = x(\widehat{\theta}_n)$) ensures that $\langle \overline{p}_n, p_{n+1} \rangle_{\underline{\theta}_n} \leq 0$.

As usual, we note that $W_2^2(\widehat{\theta}_{n+1}, \widehat{E}) \leq W_2^2(\widehat{\theta}_{n+1}, \underline{\theta}_n)$ which satisfies:

$$W_2^2\left(\widehat{\theta}_{n+1}, \underline{\theta}_n\right) = \int_{(X \times \mathcal{X})^2} \|x - z\|^2 \, \mathrm{d}\widehat{\theta}_{n+1} \otimes \underline{\theta}_n = \int_{X \times \mathcal{X}} \left\|(\overline{x}_{n+1}, \overline{y}_{n+1}) - z\right\|^2 \mathrm{d}\underline{\theta}_n(z)$$

$$= \int_{X \times \mathcal{X}} \left\| \frac{n}{n+1}(\overline{x}_n, \overline{y}_n) + \frac{1}{n+1}(x_{n+1}, y_{n+1}) - z \right\|^2 \mathrm{d}\underline{\theta}_n(z)$$

$$= \left(\frac{n}{n+1}\right)^2 \int_{X \times \mathcal{X}} \|(\overline{x}_n, \overline{y}_n) - z\|^2 \, \mathrm{d}\underline{\theta}_n(z)$$

$$+ \left(\frac{1}{n+1}\right)^2 \int_{X \times \mathcal{X}} \|(x_{n+1}, y_{n+1}) - z\|^2 \, \mathrm{d}\underline{\theta}_n(z)$$

$$+ 2\frac{n}{(n+1)^2} \int_{X \times \mathcal{X}} \langle (\overline{x}_n, \overline{y}_n) - z, (x_{n+1}, y_{n+1} - z) \rangle \, \mathrm{d}\underline{\theta}_n(z).$$

Therefore,

$$W_2^2\left(\widehat{\theta}_{n+1}, \underline{\theta}_n\right) = \left(\frac{n}{n+1}\right)^2 W_2^2\left(\widehat{\theta}_n, \underline{\theta}_n\right) + \left(\frac{1}{n+1}\right)^2 W_2^2(\theta_{n+1}, \underline{\theta}_n)$$

$$+ 2\frac{n}{(n+1)^2} \langle \overline{p}_n, p_{n+1} \rangle_{\underline{\theta}_n} \leq \left(\frac{n}{n+1}\right)^2 W_2^2\left(\widehat{\theta}_n, \widehat{E}\right) + \left(\frac{K}{n+1}\right)^2.$$

We conclude by induction over $n \in \mathbb{N}$.

We sketch the proof of the necessary part. Conclusions of Lemma 8 (delayed to Appendix) hold in $\widehat{\Gamma}$ and the proof of the first two points are identical. Hence it remains to prove the third point, i.e. that a set which is not a $\widehat{B}$-set has a secondary point (see Definition 4 also in Appendix). Let $\overline{\theta}$ be not in $\widehat{E}$, $\underline{\theta}$ one of its projection on $\widehat{E}$, and $\overline{p} \in \mathrm{NC}_{\widehat{E}}^g(\underline{\theta})$ the associated proximal normals such that:

$$\forall x \in X, \exists y \in \mathcal{X}, \langle \underline{p}, p(x, y) \rangle_{\underline{\theta}} = \int_{X \times \mathcal{X}} \langle \overline{p}(z), z - (x, y) \rangle \mathrm{d}\underline{\theta} > 0.$$

Sion's theorem implies the existence of $\delta > 0$ and $y \in \mathcal{X}$ such that for every $x \in X$, $\langle \underline{p}, p(x, y) \rangle_{\underline{\theta}} \geq \delta$. If we denote by $\theta_\lambda = (\mathrm{Id}, \sigma_\lambda) \sharp \underline{\theta}_n \otimes \theta_{n+1}$ then using the same argument as in the proof of Lemma 6, we show that

$$W_2(\overline{\theta}, \theta_\lambda) \leq W_2(\overline{\theta}, \underline{\theta}) + \frac{K\lambda^2 - 2\lambda\delta}{2W_2(\overline{\theta}, \underline{\theta})} \leq W_2(\overline{\theta}, \underline{\theta}) - \frac{\lambda\delta}{2W_2(\overline{\theta}, \underline{\theta})},$$

for λ small enough. Hence, $\underline{\theta}$ is a secondary point. $\qquad\square$

The following Theorem is the characterization of displacement convex approachable sets.

Theorem 5 *A displacement convex set $\widehat{C}$ is approachable by Player 1 in $\widehat{\Gamma}$ if and only if*

$$\forall\, y \in \mathcal{X}, \exists\, x \in X,\ \theta(x,y) = \delta_x \otimes \delta_y \in \widehat{C}.$$

The proof is based on the following lemma that extends to Wasserstein space the usual characterization of the projection on a convex set in an Euclidian space .

Lemma 6 *Let X be a compact subset of $\mathbb{R}^N$ and A be a displacement convex subset of $\Delta(X)$. Fix $\underline{\theta} \in A$. Then for all $\overline{p} \in \mathrm{NC}_A^g(\underline{\mu})$ and all $\theta_1 \in A$ we have*

$$\forall p \in \mathcal{P}(\underline{\theta}, \theta_1),\ \int_{\mathbb{R}^N} \langle \overline{p}(x), p(x) \rangle \mathrm{d}\underline{\theta}(x) := \langle \overline{p}, p \rangle_{\underline{\theta}} \leq 0. \tag{3}$$

Proof: Let us consider $\underline{\theta}, \theta_0 \in A$ and $\overline{p} \in NP_A^g(\underline{\theta})$. We denote by $\theta \notin A$ the measure outside A and $\gamma \in \Pi(\underline{\theta}, \theta)$ the optimal plan given by the definition of proximal gradient normals. Define $\gamma' = T\sharp\gamma$ where $T : (x,y) \mapsto (y,x)$ so that γ' is obviously an optimal plan from θ to $\underline{\theta}$.

Let $\widetilde{\gamma} \in \Pi(\underline{\theta}, \theta_0)$ be an optimal plan from $\underline{\theta}$ to θ_0 and for any $\lambda \in [0,1]$ we define $\theta_\lambda := \sigma_\lambda \sharp \widetilde{\gamma}$ and $\widetilde{\gamma}_\lambda = (\mathrm{Id}, \sigma_\lambda)\sharp\widetilde{\gamma}$ which belongs respectively to the displacement convex set A and to $\Pi(\underline{\theta}, \theta_\lambda)$.

By the disintegration of measure theorem for any $y \in \mathbb{R}^N$ there exists a probability measure $\widetilde{\gamma}_{\lambda,y}$ on $\mathbb{R}^N$ such that $\widetilde{\gamma}_\lambda = \int_{\mathbb{R}^N}(\delta_y \otimes \widetilde{\gamma}_{\lambda,y})\underline{\theta}(\mathrm{d}y)$ which means that for any continuous bounded function $u(y,z) : \mathbb{R}^{2N} \mapsto \mathbb{R}$

$$\int_{\mathbb{R}^{2N}} u(y,z)\widetilde{\gamma}_\lambda(\mathrm{d}y,\mathrm{d}z) = \int_{\mathbb{R}^N}\left[\int_{\mathbb{R}^N} u(y,z)\widetilde{\gamma}_{\lambda,y}(\mathrm{d}z)\right]\underline{\theta}(\mathrm{d}y)$$

We define $\widehat{\gamma} \in \Pi(\theta, \theta_\lambda)$ by:

$$\forall\phi \in C_b, \int_{\mathbb{R}^{2N}}\phi\mathrm{d}\widehat{\gamma} = \int_{\mathbb{R}^{3N}}\phi(x,z)\widetilde{\gamma}_{\lambda,y}(\mathrm{d}z)\gamma'(\mathrm{d}x,\mathrm{d}y).$$

Since $\theta_\lambda \in A$ and $\widehat{\gamma} \in \Pi(\theta, \theta_\lambda)$, we obtain:

$$W_2^2(\theta, \underline{\theta}) \leq W_2^2(\theta, \theta_\lambda) \leq \int_{\mathbb{R}^{2N}}\|x-z\|^2\mathrm{d}\widehat{\gamma}$$

$$= \int_{\mathbb{R}^{3N}}\|x-z\|^2\widetilde{\gamma}_{\lambda,y}(\mathrm{d}z)\gamma'(\mathrm{d}x,\mathrm{d}y)$$

$$= \int_{\mathbb{R}^{3N}}\|x-y\|^2\widetilde{\gamma}_{\lambda,y}(\mathrm{d}z)\gamma'(\mathrm{d}x,\mathrm{d}y) + 2\int_{\mathbb{R}^{3N}}\langle x-y, y-z\rangle\widetilde{\gamma}_{\lambda,y}(\mathrm{d}z)\gamma'(\mathrm{d}x,\mathrm{d}y)$$

$$+ \int_{\mathbb{R}^{3N}}\|y-z\|^2\widetilde{\gamma}_{\lambda,y}(\mathrm{d}z)\gamma'(\mathrm{d}x,\mathrm{d}y) = a+b+c$$

where a, b and c denote respectively the three integral terms in the above equality. It remains to estimate the three terms a, b and c.

$$a = \int_{\mathbb{R}^{2N}}\|x-y\|^2\gamma'(\mathrm{d}x,\mathrm{d}y) = W_2^2(\theta, \underline{\theta}).$$

$$b = 2\int_{\mathbb{R}^{2N}} \left\langle x - y, \int_{\mathbb{R}^N} (y - z)\widetilde{\gamma}_{\lambda,y}(\mathrm{d}z) \right\rangle \gamma'(\mathrm{d}x, \mathrm{d}y)$$

$$= 2\int_{\mathbb{R}^{2N}} \left\langle y - x, \int_{\mathbb{R}^N} (x - z)\widetilde{\gamma}_{\lambda,x}(\mathrm{d}z) \right\rangle \gamma(\mathrm{d}x, \mathrm{d}y) \quad \text{(by definition de } \gamma')$$

$$= -2\int_{\mathbb{R}^N} \left\langle \overline{p}(x), \int_{\mathbb{R}^N} (x - z)\widetilde{\gamma}_{\lambda,x}(\mathrm{d}z) \right\rangle \underline{\theta}(\mathrm{d}x) \quad \text{(from the definition of } \overline{p})$$

$$= -2\int_{\mathbb{R}^{2N}} \langle \overline{p}(x), x - z \rangle \, \widetilde{\gamma}_{\lambda,x}(\mathrm{d}z)\underline{\theta}(\mathrm{d}x)$$

$$= -2\int_{\mathbb{R}^{2N}} \langle \overline{p}(x), x - z \rangle \, \widetilde{\gamma}_{\lambda}(\mathrm{d}x, \mathrm{d}z) \quad \text{(by the desintegration formula)}$$

$$= -2\int_{\mathbb{R}^{2N}} \langle \overline{p}(x), x - [(1 - \lambda)x + \lambda z] \rangle \, \widetilde{\gamma}(\mathrm{d}x, \mathrm{d}z) \quad \text{(by definition of } \widetilde{\gamma}_{\lambda})$$

$$= -2\lambda\int_{\mathbb{R}^{2N}} \langle \overline{p}(x), x - z \rangle \, \widetilde{\gamma}(\mathrm{d}x, \mathrm{d}z) = -2\lambda\int_{\mathbb{R}^{2N}} \langle \overline{p}(x), p(x) \rangle \, \underline{\theta}(\mathrm{d}x).$$

And this holds for any $p \in \mathcal{P}(\underline{\theta}, \theta_0)$.

The disintegration of measure formula together with the definition of $\widetilde{\gamma}$ yield

$$c = \int_{\mathbb{R}^{2N}} \left\| y - [(1 - \lambda)y + \lambda z] \right\|^2 \mathrm{d}\widetilde{\gamma}(y, z) = \lambda^2 \int_{\mathbb{R}^{2N}} \left\| y - z \right\|^2 \mathrm{d}\widetilde{\gamma}(y, z)$$

hence $c = \lambda^2 W_2^2(\underline{\theta}, \theta_0)$.

Summarizing our estimates, we have obtained

$$W_2^2(\theta, \underline{\theta}) \le W_2^2(\theta, \underline{\theta}) - \lambda \int_{\mathbb{R}^{2N}} 2 < \overline{p}(x), p(x) > \underline{\theta}(\mathrm{d}x) + \lambda^2 W_2^2(\underline{\theta}, \theta_0).$$

Thus for any $\lambda \in (0, 1)$,

$$0 \le \lambda^2 W(\underline{\theta}, \theta_1) - 2\lambda \int_{\mathbb{R}^{2N}} < \overline{p}(x), p(x) > \underline{\theta}(\mathrm{d}x).$$

Dividing firstly by $\lambda > 0$ and letting secondly λ tend to 0^+, this gives the wished conclusion. $\qquad\square$

Proof of Theorem 5. Let θ be any measure not in $\widetilde{E}$ and denote by $\underline{\theta} \in \Pi_{\widehat{C}}(\theta)$ any of its projection and $\overline{p} \in NP_{\widehat{C}}(\underline{\theta})$, associated to some $\overline{\gamma} \in \Pi(\underline{\theta}, \theta)$, any proximal normal. For every $x \in X$ and $y \in \mathcal{X}$ the only optimal plan from $\underline{\theta}$ to $\delta_x \otimes \delta_y$ is $\widehat{\gamma} = \underline{\theta} \otimes (\delta_x \otimes \delta_y)$.

The function $h : X \times \mathcal{X}$ defined by

$$h(x, y) = \int_{(X \times \mathcal{X})^2} < \underline{p}(u), u - v > \mathrm{d}\widehat{\gamma}(x, y)$$

is affine in both of its variable since:

$$h(x, y) = \int_{X \times \mathcal{X}} \langle \overline{p}(u), u \rangle \, \mathrm{d}\underline{\theta}(u) - \left\langle \int_{X \times \mathcal{X}} \overline{p}(u)\mathrm{d}\underline{\theta}(u), (x, y) \right\rangle = \overline{z} - \langle z, (x, y) \rangle,$$

$$\text{where } \overline{z} = \int_{X \times \mathcal{X}} \langle \overline{p}(u), u \rangle \mathrm{d}\underline{\theta}(u) \text{ and } z = \int_{X \times \mathcal{X}} \overline{p}(u) d\underline{\theta}(u).$$

Since for every $y \in \mathcal{X}$, there exists $x \in X$ such that $\delta_x \otimes \delta_y \in C$, Proposition 6 implies that for every $y \in \mathcal{X}$ there exists $x \in X$ such that $h(x, y) \leq 0$. X and $\mathcal{X}$ are compact sets, therefore Sion's theorem implies that there exists $x \in X$ such that $h(x, y)$ for every $y \in \mathcal{X}$. Hence $\widehat{C}$ is a $\widehat{B}$-set and is approachable by Player 1.

Reciprocally, assume that there exists $y \in \mathcal{X}$ such that $\delta_x \otimes \delta_y \notin \widehat{C}$ for every $x \in X$. Since X is compact, there exists $\eta > 0$ such that $\inf_{x \in X} W_2 \left(\delta_x \otimes \delta_y, \widehat{C} \right) \geq \eta$. The strategy of Player 2 that consists of playing at each stage δ_y ensures that $\widehat{\theta}_n = \delta_{\overline{x}_n} \otimes \delta_y$ is always at, at least, $\delta > 0$ from $\widehat{C}$. Therefore it is not approachable by Player 1. $\quad\square$

Concluding remarks

Recall that action spaces in $\widetilde{\Gamma}$ (resp. $\widehat{\Gamma}$) are $\Delta(X)$ and $\Delta(\mathcal{X})$ (resp. X and $\mathcal{X}$). Assume now that in $\widetilde{\Gamma}$ players are restricted to X and $\mathcal{X}$; then a $\widetilde{B}$-set should satisfy:

$$\forall \underline{\theta} \in \widetilde{E}, \, \forall \phi \in \mathrm{NC}_{\widetilde{E}}^p(\underline{\theta}), \, \exists x \in X, \, \forall y \in \mathcal{X}, \, \int_{X \times \mathcal{X}} \phi \, \mathrm{d} \left(\underline{\theta} - \delta_x \otimes \delta_y \right) \leq 0.$$

The proof of the sufficient part of Theorem 2 does not change when we add this assumption, thus a $\widetilde{B}$-set is still approachable. However, both the proof of the necessary part of Theorem 2 and the proof of Theorem 3 are no longer valid (due to the lack of linearity).

Similarly, assume that in $\widehat{\Gamma}$ players can choose action in $\Delta(X)$ and $\Delta(\mathcal{X})$ and, at stage $n \in \mathbb{N}$, the outcome is $\theta_n = \mathbf{x}_n \otimes \boldsymbol{\xi}_n$. Strictly speaking, given such outcomes that might not be absolutely continuous with respect to $\boldsymbol{\lambda}$, the sequence of interpolation $\widehat{\theta}_n$ may not be unique. However, we can assume that the game begins at stage 2 and that $\theta_1 = \boldsymbol{\lambda}/\boldsymbol{\lambda}(X \times \mathcal{X})$; then, see e.g. Villani [21], Proposition 5.9, $\widehat{\theta}_2 \ll \boldsymbol{\lambda}$ and is unique. By induction, the sequence of $\widehat{\theta}_n$ is unique. Once again, using the same proof, we can show that a $\widehat{B}$-set is displacement approachable, but we cannot extends the necessary part nor the characterization of displacement approachable convex sets.

A   Proof of Proposition 1

Let us first state and prove the following (implicitly stated) lemma:

Lemma 7 *If the two functions ρ and s are linear both in $x \in \Delta(I)$ and $\mu \in \mathcal{S}$, then the multivalued mapping $(x, \mu) \mapsto P(x, \mu) = \{\rho(x, y); \, y \in \mathbf{s}^{-1}(\mu)\}$ is a L-Lipschitzian convex hull.*

Proof: Since the graph of $\mathbf{s}^{-1}$ is a polytope of $\mathbb{R}^{SI} \times \mathbb{R}^J$, there exists a finite family of (so called) *extreme points* functions $\{y_\kappa(\cdot); \, \kappa \in \mathcal{K}\}$ from $\mathcal{S}$ into $\Delta(J)$ that are all piecewise

linear and continuous (thus Lipschitzian) such that $\mathbf{s}^{-1}(\mu) = \operatorname{co}\{y_\xi(\mu); \ \kappa \in \mathcal{K}\}$, for every $\mu \in \mathcal{S}$. Since $\rho(x, \cdot)$ is linear on $\Delta(J)$:

$$P(x, \mu) = \left\{\rho(x, y); \ y \in \mathbf{s}^{-1}(\mu)\right\} = \left\{\rho(x, y); \ y \in \operatorname{co}\{y_\kappa(\mu), \kappa \in \mathcal{K}\}\right\} = \operatorname{co}\left\{\rho(x, y_\kappa(\mu)); \ \kappa \in \mathcal{K}\right\}.$$

Therefore P is indeed a L-Lipschitzian convex hull. □

We now turn to the actual proof of Proposition 1:

Proof of Proposition 1

The third point is obvious, so we only need to prove that if $\widetilde{E} \subset \Delta(X \times \mathcal{X})$ is approachable in $\widetilde{\Gamma}$ then $\rho(\widetilde{E}) \subset \mathbb{R}^k$ is approachable in Γ (see **part 1**) and that if $E \subset \mathbb{R}^k$ is approachable in Γ then $\rho^{-1}(E) \subset \Delta(X \times \mathcal{X})$ is also approachable (see **part 2**). The remaining easily follows from the fact that $\rho(\rho^{-1}(E)) \subset E$.

part 1: The proof consists in two steps. First, we link the Wasserstein distance between two probability measures $\overline{\theta}, \underline{\theta} \in \Delta(X \times \mathcal{X})$ and the distance between the two sets $\rho(\overline{\theta}) \subset \mathbb{R}^k$ and $\rho(\underline{\theta}) \subset \mathbb{R}^k$. We will prove this step with the use of the 1-Wasserstein distance defined below. In the second step, we transform a strategy in $\widetilde{\Gamma}$ into a strategy in Γ^p.

<u>Step 1:</u> The 1-Wasserstein distance between μ and ν in $\Delta(X)$ is defined by:

$$W_1(\overline{\theta}, \underline{\theta}) = \inf_{\gamma \in \Pi(\overline{\theta}, \underline{\theta})} \int_X \|x - y\| \mathrm{d}\gamma(x, y) = \sup_{\phi \in \operatorname{Lip}_1(X, \mathbb{R})} \int_X \phi \ \mathrm{d}\overline{\theta} - \mathrm{d}\underline{\theta} = \inf_{U \sim \overline{\theta}, V \sim \underline{\theta}} \mathbb{E}[\|U - V\|],$$

where $\operatorname{Lip}_1(X, \mathbb{R})$ is the set of 1-Lipschitzian functions from X to $\mathbb{R}$. Jensen's inequality and the probabilistic interpretation imply that $W_1(\overline{\theta}, \underline{\theta}) \leq W_2(\overline{\theta}, \underline{\theta})$.

Let P is a L-Lipschitzian convex hull, then since $P(x, \xi) = \operatorname{co}\{p_\kappa(x, \xi); \kappa \in \mathcal{K}\}$ where every p_κ is L-Lipschitzian, for every $\overline{\theta} \in \Delta(X \times \mathcal{X})$, by convexity of the integral (see e.g. Klein and Thompson [13], Theorem 18.1.19):

$$\int_{X \times \mathcal{X}} P(x, \xi) \mathrm{d}\overline{\theta} = \int_{X \times \mathcal{X}} \operatorname{co}\{p_\kappa(x, \xi); \ \kappa \in \mathcal{K}\} \mathrm{d}\overline{\theta} = \operatorname{co}\left\{\int_{X \times \mathcal{X}} p_\kappa(x, \xi) \mathrm{d}\overline{\theta}; \ \kappa \in \mathcal{K}\right\}.$$

The mapping p_κ is L-Lipschitzian, so $d\left(\int_{X \times \mathcal{X}} p_\kappa(x, \xi) \mathrm{d}\overline{\theta}, \rho(\underline{\theta})\right) \leq \sqrt{k}L\varepsilon$ and since the set $\rho(\underline{\theta})$ is convex, $d\left(\rho(\overline{\theta}), \rho(\underline{\theta})\right) \leq \sqrt{k}L\varepsilon$. Therefore,

$$W_2\left(\overline{\theta}, \widetilde{E}\right) \leq \varepsilon \implies \sup_{z \in \rho(\overline{\theta})} d(z, \rho(\widetilde{E})) \leq \sqrt{k}L\varepsilon.$$

<u>Step 2:</u> This step transforms a strategy in $\widetilde{\Gamma}$ into a strategy in Γ and is quite standard in games with partial monitoring (see e.g. Lugosi, Mannor and Stoltz [16]); its proof, which relies deeply Hoeffding-Azuma's [12, 5] inequality, is therefore only sketched.

Let $\widetilde{\sigma}$ be a strategy of Player 1 that approaches (up to $\varepsilon > 0$) a set $\widetilde{E} \subset \Delta(X \times \mathcal{X})$. This strategy cannot be directly played in Γ in order to approach $\rho(\widetilde{E})$ for two reasons:

1) in Γ, Player 1 chooses an action $i \in I$ and not some $\mathbf{x} \in \Delta(\Delta(I))$;

2) at stage n in Γ, the flag $\mu_n = \mathbf{s}(i_n, j_n)$ is not observed, but only a signal s_n whose law is $s(i_n, j_n)$.

The usual trick is to divide $\mathbb{N}$ into blocks of length $N \in \mathbb{N}$ – where N is big enough. The n-th block in Γ will correspond to the n-th stage in $\widetilde{\Gamma}$ and σ is defined inductively. Assume that $\widetilde{\sigma}$ dictates to play $\mathbf{x}_n \in \Delta(X)$ at the n-th stage of $\widetilde{\Gamma}$. In Γ and independently at every stage t of the n-th block, with probability η the action i_t is chosen uniformly over I, and with probability $1 - \eta$ accordingly to $\mathbb{E}_{\mathbf{x}_n} \in X$.

The stages where i_t was chosen uniformly allow to build an unbiased estimator that will be arbitrarily close (if N is big enough) to $\overline{\mu}_n$, the average flag during the n-th block. Since the choice of actions are independent, it is easy to show that the average payoff on the n-th block is arbitrarily close to the η-neighborhood of $P(\mathbb{E}_{\mathbf{x}_n}, \overline{\mu}_n) = \rho(E_{\mathbf{x}_n}, \delta_{\overline{\mu}_n})$. So it is enough to act as if Player 2's action in $\widetilde{\Gamma}$ was $\delta_{\overline{\mu}_n}$.

In order to obtain the almost surely convergence, we can use a classical *doubling trick* argument (see e.g. Sorin [19]), which consists in a concatenation of these strategies with increasing N and decreasing η.

Part 2: Assume that $E \subset \mathbb{R}^k$ is approachable in Γ by Player 1. Consider the game where Player 1 observes in addition $\mu_n = \mathbf{s}(j_n)$ and his payoff is $\mathbb{E}_{x_n}[\rho(i_n, j_n)]$ where x_n is the law of i_n. This new game is *easier* for Player 1 because he has more information and actions, hence he can still approach $E \subset \mathbb{R}^k$. Since $P(x, \cdot)$ is convex, allowing Player 2 to play any action in $\Delta(\mathcal{X})$ does not make the game harder for Player 1. Thus we can assume that at stage $n \in \mathbb{N}$, Player 1 observes $\boldsymbol{\xi}_n \in \Delta(\mathcal{X}) = \Delta(\mathcal{S})$, that he plays deterministically $\mathbf{x}_n \in \Delta(X) = \Delta(\Delta(I))$ and that his payoff belongs to $\rho(\mathbf{x}_n \otimes \boldsymbol{\xi}_n)$. We call this new game by Γ^d.

The fact that E is approachable in Γ^d implies that for every ε there exists a strategy σ_ε in Γ^d and $N_\varepsilon \in \mathbb{N}$ such that for every $n \geq N_\varepsilon$ and strategy τ of Player 2:

$$d\left(\frac{\sum_{m=1}^n \rho(\mathbf{x}_m \otimes \boldsymbol{\xi}_m)}{n}, E\right) := \sup\left\{d(z, E); \ z \in \frac{\sum_{m=1}^n \rho(\mathbf{x}_m \otimes \boldsymbol{\xi}_m)}{n}\right\} \leq \varepsilon. \quad (4)$$

If we denote as before $\theta_n = \mathbf{x}_n \otimes \boldsymbol{\xi}_n \in \Delta(X \times \mathcal{X})$ then equation (4) becomes:

$$\forall \varepsilon > 0, \ \exists N_\varepsilon \in \mathbb{N}, \ \forall n \geq N_\varepsilon, \ \overline{\theta}_n \in \rho^{-1}(E^\varepsilon).$$

Let us define similarly $\rho^{-1}(E)^\delta = \left\{\theta \in \Delta(X \times \mathcal{X}); \ W_2\left(\theta, \widetilde{E}\right) \leq \delta\right\}$. Since the sequence of compact sets $\rho^{-1}(E^\varepsilon)$ converges (as ε converges to zero) to $\rho^{-1}(E)$, for every $\delta > 0$, there exists $\underline{\varepsilon}$ such that for every $0 < \varepsilon \leq \underline{\varepsilon}$, $\rho^{-1}(E^\varepsilon) \subset \rho^{-1}(E)^\delta$. Therefore, for every $\delta > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$ and every strategy τ of Player 2, $\overline{\theta}_n$ belongs to $\rho^{-1}(E)^\delta$. Thus $\rho^{-1}(E)$ is approachable by Player 1. $\qquad \square$

B   Proof of the necessary part of Theorem 2

The necessary part of Theorem 2 is an immediate consequence of Lemma 8, greatly inspired from Spinat [20]; it requires the following definition.

Definition 4 *A point $\theta \in \Delta(X \times \mathcal{X})$ is δ-secondary for $\widetilde{E}$ if there exists a corresponding couple: a point $\boldsymbol{\xi} \in \Delta(\mathcal{X})$ and a continuous function $\lambda : \Delta(X) \to (0,1]$ such that $\min_{\mathbf{x} \in \Delta(X)} W_2\left(\lambda(\mathbf{x})\theta + (1 - \lambda(\mathbf{x}))\mathbf{x} \otimes \boldsymbol{\xi}, \widetilde{E}\right) \geq \delta$. A point θ is secondary to $\widetilde{E}$ if there exists $\delta > 0$ such that x is δ-secondary to $\widetilde{E}$.*

We denote by $\mathcal{PP}(\widetilde{E}) \subset \widetilde{E}$ the subset of primary point to $\widetilde{E}$ (i.e. points of $\widetilde{E}$ that are not secondary).

Lemma 8 (Spinat [20])

   i) *Any approachable compact set contains a minimal approachable set;*

   ii) *A minimal approachable set is a fixed point of $\mathcal{PP}$;*

   iii) *A fixed point of $\mathcal{PP}$ is a $\widetilde{B}$-set.*

Proof: *i)* Let $\mathcal{B} = \left\{ \widetilde{B} \subset \widetilde{E} \middle| \widetilde{B} \text{ is an approachable compact set} \right\}$ be a nonempty family ordered by inclusion. Every fully ordered subset of $\mathcal{B}$ has a minorant $\underline{\widetilde{B}}$ (the intersection of every elements of the subset) that belongs to $\mathcal{B}$ since it is an approachable compact subset of $\widetilde{E}$. Thus Zorn's lemma yields that $\mathcal{B}$ contains at least one minimal element.

ii) We claim that if $\widetilde{E}$ is approachable then so is $\mathcal{PP}(\widetilde{E})$, hence a minimal approachable set is necessarily a fixed point of $\mathcal{PP}$. Indeed, if θ is δ-secondary there exists an open neighborhood V of θ such that every point of V is $\delta/2$-secondary, because of the continuity of W_2. Hence $\mathcal{PP}(\widetilde{E})$ is a compact subset of $\widetilde{E}$.

Let θ_0 be a δ-secondary point of an approachable set $\widetilde{E}$ and $\boldsymbol{\xi}, \lambda$ the associated couple given in Definition 4. Let $\varepsilon < \delta/4$ and consider σ a strategy of Player 1 that ensures that $\overline{\theta}_n$ is, after some stage $N \in \mathbb{N}$, closer than ε to $\widetilde{E}$. We will show that $\overline{\theta}_n$ must be close to θ_0 only a finite number of times; so Player 1 can approach $\widetilde{E} \backslash \{\theta\}$. Indeed, assume that there exists a stage $m \in \mathbb{N}$ such that $W_2(\overline{\theta}_m, \theta_0) \leq \delta/4$ and consider the strategy of Player 2 that consists in playing repeatedly $\boldsymbol{\xi}$ from this stage on. It is clear that (if m is big enough) after some stage $\overline{\theta}$ will be $\delta/2$-closed to $\lambda(\overline{\mathbf{x}}_{n,m})\theta_0 + (1 - \lambda(\overline{\mathbf{x}}_{n,m}))\overline{\mathbf{x}}_{n,m} \otimes \boldsymbol{\xi}$ where $\overline{\mathbf{x}}_{n,m}$ is the average action played by Player 1 between stage m and $m+n$. Therefore $W_2\left(\overline{\theta}_n, \widetilde{E}\right) \geq \delta/2 > \varepsilon$ and since $W_2(\overline{\theta}_n, \theta_0)$ can be bigger than $\delta/4$ only a finite number of times, the strategy of Player 1 approaches $\widetilde{E} \backslash \{\theta\}$. This is true for any secondary point, so Player 1 can approach $\mathcal{PP}(\widetilde{E})$.

iii) Assume that $\widetilde{E}$ is not a $\widetilde{B}$-set: there exists $\overline{\theta} \notin \widetilde{E}$ such that for any projection $\underline{\theta} \in \Pi_{\widetilde{E}}(\overline{\theta})$, any $\phi \in \Phi(\underline{\theta}, \overline{\theta})$ and any $\mathbf{x} \in \Delta(X)$, there exists $\boldsymbol{\xi} \in \Delta(\mathcal{X})$ such that $\int_{X \times \mathcal{X}} \phi \mathrm{d}(\underline{\theta} - \mathbf{x} \otimes \boldsymbol{\xi}) > 0$. This last expression is linear both in $\mathbf{x}$ and $\boldsymbol{\xi}$, so Von Neumann's minmax theorem imply that there exists $\boldsymbol{\xi}(= \boldsymbol{\xi}(\underline{\theta}, \phi))$ and δ such that $\int_{X \times \mathcal{X}} \phi \mathrm{d}(\underline{\theta} - \mathbf{x} \otimes \boldsymbol{\xi}) \geq \delta > 0$, for every $\mathbf{x} \in \Delta(X)$.

We can assume that $\overline{\theta} \in \Delta_0(X \times \mathcal{X})$. Otherwise, let $(\overline{\theta}_n \in \Delta_{\frac{1}{n}}(X \times \mathcal{X}))_{n \in \mathbb{N}}$ be a sequence of measures that converges to $\overline{\theta}$, $(\underline{\theta}_n)_{n \in \mathbb{N}}$ a sequence of projection of $\overline{\theta}_n$ onto $\widetilde{E}$ and $\phi_0^n \in \Phi(\overline{\theta}_n, \underline{\theta}_n)$. Up to two extractions, we can assume that $\underline{\theta}_n$ converges to $\underline{\theta}_0$ a

projection of $\overline{\theta}$ and ϕ_0^n converges to $\phi_0 \in \Phi(\overline{\theta}, \underline{\theta}_0)$. Therefore, for n big enough and for every $\mathbf{x} \in \Delta(X)$,

$$0 < \frac{\delta}{2} \leq \int_{X \times \mathcal{X}} \phi_0^n \mathrm{d}\left(\underline{\theta}_n - \mathbf{x} \otimes \boldsymbol{\xi}(\underline{\theta}_0, \phi_0)\right)$$

since the right member converges to $\int_{X \times \mathcal{X}} \phi_0 \mathrm{d}\left(\underline{\theta}_0 - \mathbf{x} \otimes \boldsymbol{\xi}(\underline{\theta}_0, \phi_0)\right) \geq \delta$.

For every $\lambda \in [0, 1]$ and $\mathbf{x} \in \Delta(X)$, we denote by $\phi_{\lambda, \mathbf{x}}$ the unique (we assumed that $\overline{\theta} \ll_0$) Kantorovitch potential such that:

$$W_2^2\left((1 - \lambda)\underline{\theta} + \lambda \mathbf{x} \otimes \boldsymbol{\xi}, \overline{\theta}\right) = \int_{X \times \mathcal{X}} \phi_{\lambda, \mathbf{x}} \mathrm{d}\left((1 - \lambda)\underline{\theta} + \lambda \mathbf{x} \otimes \boldsymbol{\xi}\right) + \int_{X \times \mathcal{X}} \phi_{\lambda, \mathbf{x}}^* \mathrm{d}\overline{\theta}$$

$$= \int_{X \times \mathcal{X}} \phi_{\lambda, \mathbf{x}} \mathrm{d}\underline{\theta} + \int_{X \times \mathcal{X}} \phi_{\lambda, \mathbf{x}}^* \mathrm{d}\overline{\theta} - \lambda \int_{X \times \mathcal{X}} \phi_{\lambda, \mathbf{x}} \mathrm{d}(\underline{\theta} - \mathbf{x} \otimes \boldsymbol{\xi}).$$

Since $(\lambda, \mathbf{x}) \mapsto \phi_{\lambda, \mathbf{x}}$ is continuous, $\phi_{\lambda, \mathbf{x}}$ converges to ϕ_0, for every $\mathbf{x} \in \Delta(X)$ which is compact. Hence there exists $\underline{\lambda} \in (0, 1]$ such that:

$$\left| \int_{X \times \mathcal{X}} (\phi_{\lambda, \mathbf{x}} - \phi_0) \, \mathrm{d}(\underline{\theta} - \mathbf{x} \otimes \boldsymbol{\xi}) \right| \leq \delta/4, \quad \forall \lambda \leq \underline{\lambda}, \forall \mathbf{x} \in \Delta(X).$$

Therefore, one has $\left(W_2\left(\overline{\theta}, (1 - \underline{\lambda})\underline{\theta} + \underline{\lambda}\mathbf{x} \otimes \boldsymbol{\xi}\right)\right)^2 \leq \left(W_2(\overline{\theta}, \underline{\theta})\right)^2 - \underline{\lambda}\frac{\delta}{4}$ so

$$W_2\left(\overline{\theta}, (1 - \underline{\lambda})\underline{\theta} + \underline{\lambda}\mathbf{x} \otimes \boldsymbol{\xi}\right) \leq W_2(\overline{\theta}, \underline{\theta}) - \frac{\underline{\lambda}\delta}{8W_2(\overline{\theta}, \underline{\theta})} := W_2(\overline{\theta}, \underline{\theta}) - \eta$$

which implies that $W_2\left((1 - \underline{\lambda})\underline{\theta} + \underline{\lambda}\mathbf{x} \otimes \boldsymbol{\xi}, \widetilde{E}\right) \geq \eta$ and $\underline{\theta}$ is η-secondary to $\widetilde{E}$.

Consequently, a fixed point of $\mathcal{PP}$, i.e. a set without any secondary point, is necessary a $\widetilde{B}$-set. $\qquad \square$

References

[1] S. As Soulaimani *Viability with probabilistic knowledge of initial condition, application to optimal control*, Set-Valued Anal. **16** (2008), 1037–1060.

[2] S. As Soulaimani, M. Quincampoix, and S. Sorin *Repeated games and qualitative differential games: approachability and comparison of strategies*, SIAM J. Control Optim. **48** (2009), 2461–2479

[3] J.-P. Aubin and H. Frankowska, *Set-valued analysis*, Birkhäuser Boston Inc, 1990.

[4] R.J. Aumann and M. B. Maschler *Repeated games with incomplete information*, MIT Press, 1955.

[5] K. Azuma *Weighted sums of certain dependent random variables*, Tôhoku Math. J. **19** (1967), 357–367.

[6] D. Blackwell *An analog of the minimax theorem for vector payoffs.* Pacific J. Math., **6** (1956), 1–8.

[7] J.-M. Bony *Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier (Grenoble), **19** (1969), 277-304

[8] Y. Brenier *Décomposition polaire et réarrangement monotone des champs de vecteurs*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), 805–808.

[9] P. Cardaliaguet and M. Quincampoix *Deterministic differential games under probability knowledge of initial condition* International Game Theory Review. **10** (2008), 1-16.

[10] N. Cesa-Bianchi and G. Lugosi *Prediction, learning, and games*, Cambridge University Press, 2006.

[11] R.M. Dudley *Real analysis and probability*, Cambridge University Press, 1989.

[12] W. Hoeffding *Probability inequalities for sums of bounded random variables*, J. Amer. Statist. Assoc. **58** (1963), 13–30.

[13] E. Klein and A. Thompson *Theory of correspondences*, John Wiley & Sons Inc, 1984.

[14] E. Kohlberg *Optimal strategies in repeated games with incomplete information* Internat. J. Game Theory, **4** (1975), 7–24

[15] E. Lehrer and E. Solan *Learning to play partially-specified equilibrium*, mimeo (2007).

[16] G. Lugosi, S. Mannor, and G. Stoltz *Strategies for prediction under imperfect monitoring*, Math. Oper. Res., **33** (2008), 513–528.

[17] V. Perchet *Approachability of convex sets with partial monitoring*, J. Optim. Theory. Appl., **149** (2011), 665–677

[18] L.S. Shapley *Stochastic games*, Proc. Nat. Acad. Sci. U. S. A. **39** (1953), 1095–1100.

[19] S. Sorin *Supergames*, Game Theory and its Applications (T. Ishiishi, A. Neyman, and Y. Tauman, ed.), Academic Press, 1990, 46-63.

[20] X. Spinat *A necessary and sufficient condition for approachability*, Math. Oper. Res., **27** (2002), 31–44.

[21] C. Villani *Topics in optimal transportation*, Graduate studies in Mathematics, AMS, Vol. 58, 2003.